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A Generalized Markowitz Portfolio Selection Model with Higher Moments

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Abstract: This paper proposes a generalized Markowitz portfolio investment model via adding measures of skewness and peakedness into the original Markowitz investment model. With the third and fourth moment in the objective function, we find the magnitude of risk and shapes of the efficient frontier differ from that of the original model. And the original Markowitz model can be seen as a special case of the generalized model.

1. Introduction

No doubt, one of the important applications of quadratic programming is the Markowitz portfolio selection model (1952 and 1959) upon which modern investment theory is built. While the quadratic spatial equilibrium model (Takayama and Judge 1971) had wide applications in agriculture and energy markets (see Labys and Yang, 1997), the applications of the Markowitz portfolio model are limited mostly to financial markets and used scarcely in wine investment market (Labys et al. 1981). Perhaps it is one of the least understood models in the finance literature since his model primarily falls within the domain of operations research (Markowitz 1956). Nonetheless, the portfolio selection models have since advanced beyond its prototype (see Sharpe 1963 and 1964, Lintner 1965, Mossin 1956, Ross 1976, Markowitz and Perold 1981 and Markowitz 1987). Not surprisingly, the main focus of these models is primarily on the expansions and improvements in the mean variance space including the equivalence of the Markowitz risk minimization and Sharpe angle maximization models (Yang et al. 2002). Well-known in the literature, if stock returns follow normal distribution, mean and variance are sufficient to describe the return behavior for diversifier. More often than not, however, stock returns do not obey normality and as such higher moments may well be required for more accurate depiction of stock market. The traditional quadratic utility function may well be inadequate to describe a class of behaviors when the degree of skewness plays a role in investment. The purpose of this paper is to expand the original Markowitz model via adding the measures of skewness (third moment about the mean) and peakedness (fourth moment about the mean) as will be done in the next section. Section III provides empirical results by using five companies and compares the efficient frontiers between the original and augmented Markowitz portfolio selection models. It can be easily shown that the original Markowitz model is a special case of the generalized model. A conclusion is given in section IV.

2. An Augmented Markowitz Portfolio Investment Model

Given a security market of n stocks, the original Markowitz portfolio selection model (1952, 1956) takes the form of the following minimization model:

Minimize $\quad v = \Sigma s_{ii}\, x_i^2 + \Sigma\ \Sigma x_i x_j s_{ij}$ [1]

$x_i \quad i \in I \;\; i \in I \;\; j \in J \quad j \neq i$

Subject to $\quad \Sigma m_i x_i \geq k$ [2]

$i \in I$

$\Sigma\, x_i = 1$ [3]

$i \in I$

$x_i \geq 0 \qquad \forall i \in I$ [4]

where v is the risk represented by the weighted variance and covariance of the portfolio returns with the weight ; s_{ii} and s_{ij} are sample variance of the return of security i for each i, and covariance of returns between security i and j respectively; x_i denotes the proportion of total investment on security i; m_i is the expected return of security i or decision variable of the model; k denotes the minimum expected portfolio return reflecting investors preference on investment's rate of return, and I, J are a set of positive integers (1, ...n). The properties of the quadratic minimization problem are well travelled (Markowitz 1959), and its efficient computational algorithms are analyzed by Tucker and Dafaro (1975), Pang (1980), Schrage (1986), and Markowitz (1956, 1987).

Since the Markowitz model involves making choices in the mean-variance-covariance space, it can be reformulated as follows:

Minimize $\quad -u_1(\Sigma m_i x_i) + u_2 V$ [5]

x_i

Subject to [3] and [4]

Where u_1 and $u_2 = 1 - u_1$ represent the exogenously determined weights assigned to stock return and risk respectively. The two models are equivalent for two reasons: First, minimizing negative returns is identical to maximizing the positive returns for a given constraint set. Second, once the optimum xi's are obtained from equation 5, $\Sigma mi\ \ xi$ is a constant and as such can be removed from the objective function to the constrain set as $\Sigma mi\ xi$= k or $\Sigma mi\ \ xi$≥k for a binding constraint without affecting the optimum solutions except for the value of objective function. This is to say, for a given set of values of $u1$ and $u2$, there exists a target rate of portfolio returns $\Sigma mi\ \ xi$= k, which can be added back to the constraint sets 3 and 4 to become equation 2 , The resulting model is again the original Markowitz portfolio model. For a given set of $u1$ and $u2 = 1 - u1$, there exist a Markowitz minimization model with a target portfolio return $\Sigma mi\ \ xi$≥k. In other words, the first two moments can be formulated in the objective function and both models produce identical efficient frontier curves.

Alternatively, one can reformulated the Markowitz model as

Minimize $-\sum_{i\epsilon I} m_i \, x_i$ [6]

Subject to $v = v^*$ [7]

And [3], [4]

where v^* is evaluated at the optimality from [5]. Note that minimization of $-\Sigma m_i x_i$ subject to [3], [4] and [7] produces the identical solutions as the of minimizing $-u_1 \Sigma m_i x_i$ except for the value of objective function which is now a fraction of $\Sigma m_i \, x_i^*$ since $-u_1$ is a predetermined scalar. To follow the convention, and to allow higher moments to enter risk evaluation, we employ equations [1] through [4] to extend the original Markowitz portfolio model. In what follows, we expand the risk evaluation beyond variance –covariance space. When stock returns are normally distributed, their distribution exhibits symmetry with appropriate tail thickness. In absence of such a normality, higher moments around the mean are to be considered. Population skewness coefficient $\mathrm{E}[(x-u)^3 \,]/\, \sigma^3$ measures the asymmetry of a distribution while population kurtosis $\mathrm{E}[(x-u)^4]/\sigma^4$ measures the thickness of the tails of a distribution. To incorporate these coefficients into the risk considerations, we formulate the augmented Markowitz portfolio investment model:

Minimize $R = u_2(\sum_{i\epsilon I} s_{ii} \, x_i^2 + \sum_{i\epsilon I} \sum_{\mathrm{i\epsilon J}} \mathrm{s_{ij}} \, x_i x_j)$ [8]

$+ \; u_3(\sum_{i\epsilon I} x_i^3 \, SK_i) + u_4(\sum_{i\epsilon I} x_i^4 \, KT_i)$

Subject to $\sum_{i\epsilon I} m_i \, \overline{x_i} \geq k$ [9]

$\sum_{i\epsilon I} x_i = 1$ [10]

$x_i \geq 0$ for all $\mathrm{i}\epsilon I$ [11]

where u_2, u_3, u_4 are weights on different levels of risk such that u_1+u_2+u_3=1; SK_i is the sample coffocient of skewness of stock i; $\mathrm{KT_i}$ is the sample coefficient of kurtosis of stock i.

Note that the third moment about the mean — $SK_i = \Sigma(x_i - \bar{x})^3$/n-1/$s_i^3$ where s_i denotes sample standard deviation— can be positive (skewed to the right) or negative (skewed to the left), if there exist a handful of unusually large or small outliers. The coefficient of sample kurtosis $KT_i = (\Sigma(x_i - \bar{x})^4$/n-1/$s_i^3) - 3$ can be zero for normally distributed stock returns. It is positive for highly clustered or peaked stock returns (or leptokurtic) whereas it is negative for widely spread or thick-tailed stock returns distribution (or platykurtic). In addition, majority of stock returns in a left-skewed distribution ($Sk_i < 0$) have rate of return greater than the mean return for mean is the smallest of the three measures for central location. This property may be deemed desirable for investors who place high value on “more than average” return. The flip side is that there exists a slim chance that returns can be unusually small. The net effect depends on investor’s attitude toward risk and hence the weight placed on the risk emanated from coefficient of skewness (u_3) is unique to different investors. The converse holds for a right-skewed distribution ($SK_i > 0$).

In the case of a platykurtic returns distribution in which fat tails are present or $KT_i < 0$, we alter the sign to be positive: a positive value reflects potential risk due to relatively more large of small stock returns. In contrast, a leptokurtic distribution in which a great majority of stock returns are clustered around the mean may reflect stability due to its peakedness. Again, the weight on kurtosis, $u4$, depends on investors' utility function and all the weights on the second, third and fourth moments are predetermined before solving the augmented portfolio investment model: $u_1 + u_2 + u_3 = 1$.

3. Empirical Results

We use monthly stock prices from September 2007 through August 2008 to compute rate of return for five companies: MasterCard, IBM, J&J, McDonald, and WalMart. Mean, variance, covariance, coefficients of skewness and kurtosis are reported in Table 1. We then substitute these sample estimates and k (from 1.5% to 4.5%) into [8], [9], [10], and [11] to solve for optimum investment weights x_i for both original Markowitz model with only second moment in the objective function and augmented Markowitz model with second, third and fourth moments. For simplicity and neutrality, we $u_1 = u_2 = u_3 = \frac{1}{3}$ assume . The results are reported in Table 2.

Table 1
Descriptive Statics

Company	**MA**	**IBM**	**JNJ**	**MCD**	**WMT**
	1	**2**	**3**	**4**	**5**
Mean	0.045197848	0.004375371	0.008717499	0.015885226	0.029098589
Standard Deviation	0.131243596	0.061399467	0.037827323	0.053142803	0.040189604
Sample Variance	0.017224882	0.003769895	0.001430906	0.002824157	0.001615204
Kurtosis	1.062960722	0.990754052	1.129677383	0.709394182	1.135660187
Coefficient of Skewness	0.179168884	0.470441142	0.211762925	0.755623309	0.064618043

$\mathbf{Cov_1, 2 = 0.0011587}$ (Covariance between MA and IBM)

$\mathbf{Cov_1, 3 = 0.0012066}$ (Covariance between MA and JNJ)

$\mathbf{Cov_1, 4 = 0.0035370}$ (Covariance between MA and MCD)

$\mathbf{Cov_1, 5 = 0.0026356}$ (Covariance between MA and WMT)

$\mathbf{Cov_2, 3 = 0.0003044}$ (Covariance between IBM and JNJ)

$\mathbf{Cov_2, 4 = 0.0008814}$ (Covariance between IBM and MCD)

$\mathbf{Cov_2, 5 = 0.0000637}$ (Covariance between IBM and WMT)

$\mathbf{Cov_3, 4 = 0.0012232}$ (Covariance between JNJ and MCD)

$\mathbf{Cov_3, 5 = 0.0005912}$ (Covariance between JNJ and WMT)

$\mathbf{Cov_4, 5 = 0.0004080}$ (Covariance between MCD and WMT)

Table 2

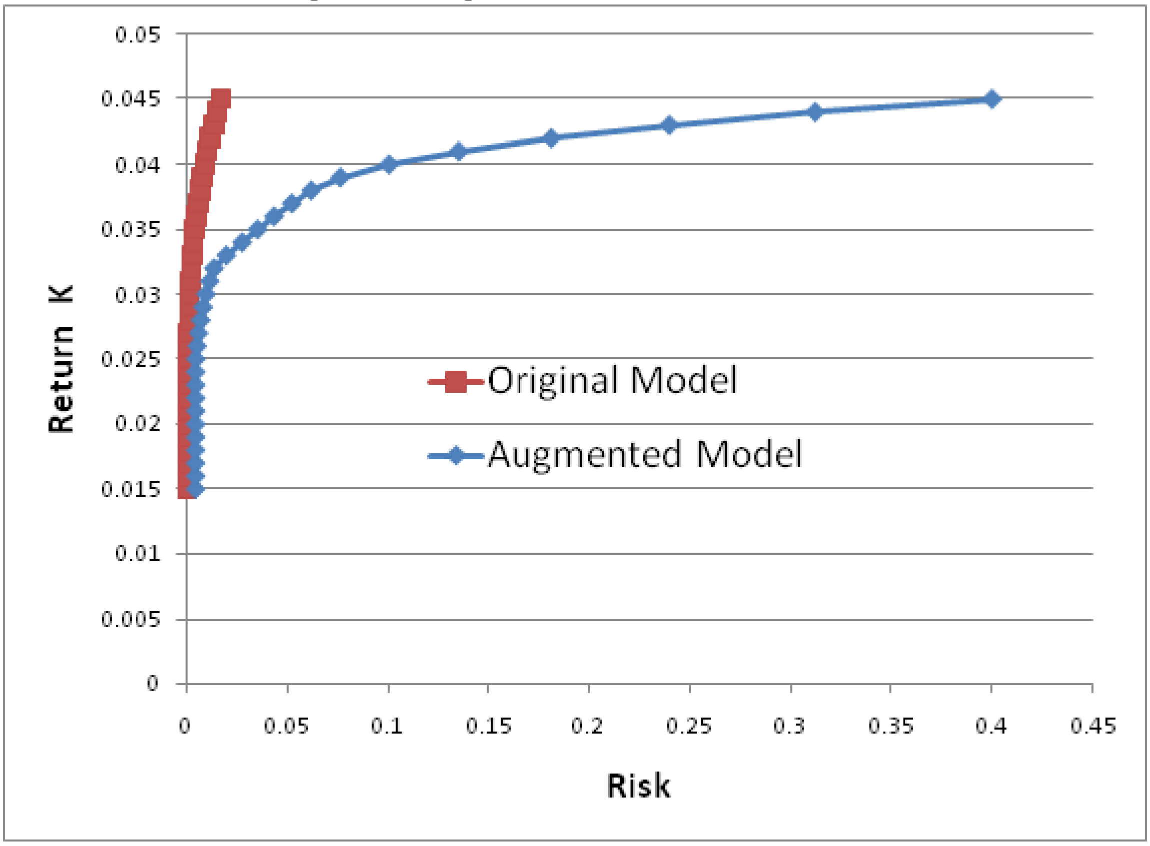
Mean and Risk Under Original and Augmented Markowitz Models

K	v	K	v+sk+kt
0.015	0.0008851094	0.015	0.005007084
0.016	0.0008851094	0.016	0.005007084
0.017	0.0008883713	0.017	0.005007084
0.018	0.0008983724	0.018	0.005007084
0.019	0.0009151138	0.019	0.005007084
0.02	0.0009385953	0.02	0.005007084
0.021	0.0009688170	0.021	0.005007084
0.022	0.0010057790	0.022	0.005007084
0.023	0.0010494810	0.023	0.005007084
0.024	0.0010999230	0.024	0.005007084
0.025	0.0011584800	0.025	0.005131101
0.026	0.0012324610	0.026	0.005679282
0.027	0.0013229430	0.027	0.006493584
0.028	0.0014395070	0.028	0.007519602
0.029	0.0015906850	0.029	0.008730243
0.03	0.0017720080	0.03	0.01018907
0.031	0.0020455100	0.031	0.01200019
0.032	0.0024237160	0.032	0.01411537
0.033	0.0029066260	0.033	0.02021092
0.034	0.0034942400	0.034	0.02804308
0.035	0.0041865580	0.035	0.03573452
0.036	0.0049835800	0.036	0.04380183
0.037	0.0058853060	0.037	0.05261469
0.038	0.0068917350	0.038	0.06247662
0.039	0.0080028690	0.039	0.07679115
0.04	0.0092187060	0.04	0.1009397
0.041	0.0105392500	0.041	0.1355897
0.042	0.0119644900	0.042	0.1816042
0.043	0.0134944400	0.043	0.2401082
0.044	0.0151290900	0.044	0.3124882
0.045	0.0168684500	0.045	0.4003928

The figures in the first two columns are from the original Markowitz model; those in the next two columns are from the generalized Markowitz model.

An examination on Table 2 indicates that efficient frontier in the mean-risk space are rather different for the augmented Markowitz model. With skewness and peakedness considered, the risk levels exceed that of the original model. As shown in Figure 1, the new efficient frontier lies to the right of that in the original model. In addition, the concavity of the conventional efficient frontier in the original Markowitz model may not be preserved as cubic and fourth power terms are added to the model. The implication can be significant: the uniqueness of capital market equilibrium may simply not occur and multiple solutions can be a distinct possibility.

Figure 1
Efficient Frontiers in Original and Augmented Markowitz Model



4. Conclusion

It is over half a century since Markowitz's seminal paper on portfolio investment model on which modern investment theories are established. The idea of diversification via negative covariance was novel and quadratic programming was in its infant stage. However, if stock returns do not follow normal distribution, the first two moments may be inadequate to describe investment behaviors. In this paper, we propose a generalized Markowitz investment model via adding degrees of skewness and peakedness of stock returns in the hope of providing a wider perspective on investment behavior. When the weights on skewness and kurtosis equal zero, i.e. , $u_3 = u_4 = 0$, the only remaining weight u_2 equals one. The generalized investment model is reduced to the original Markowitz model. This is to say, the Markowitz model is a special case of the generalized model. It is found that magnitude of the risk measure of the latter exceeds that of the former. And, shape of efficient frontier in the generalized model may not be concave as is guaranteed in the original quadratic programming model. It has important implications in the capital market equilibrium: it may not have a unique solution even with an investor's smooth convex indifference curve. The results from using five companies suggest that equilibrium in the mean-risk space may differ substantially. Finally, given the convenience and power of optimization software, computational problems are now greatly reduced at lease for medium-sized problem in the presence of high moments in stock markets.

5. Conclusions

In this paper, we investigate the two potential sources of market discipline, uninsured deposits and subordinated debt, following the conventional wisdom that these at-risk claimants have strong incentives to discipline the banks through rationing and/or pricing their credit using a sample of BHCs from 2001 to 2005, a period of stability and growth. We first test to see if the documented monitoring function holds for the more recent period in the banking industry for both uninsured deposits and subordinated debt. We then proceed to investigate the influencing effect of these disciplinary actions by depositors/creditors on bank behavior in the following year, by simply switching the traditional monitoring model. We further test the model by focusing on more specific type of reaction by depositors/creditors, namely a negative reaction such as rationing their credit (decrease in their holdings) or charging higher interest (increase in prices).

The results are not very encouraging: although we find some evidence of monitoring, especially by uninsured depositors, we don't find any evidence of any bank responses to these monitoring activities, especially those by subordinated debt holders. The only bank responses are to the changes in uninsured depositors fund levels or some to the changes in their prices but responses to price changes disappear completely when we focus specifically on the traditionally defined monitoring activities: punishing the risky, non-performing banks by rationing the credit or charging higher interest.

We conclude that high expectations from market discipline for banking system stability may be premature. There appear to be some useful signals coming from the market participants, but not strongly enough to substitute for regulatory vigilance and prompt corrective actions. The results have potentially significant and cautionary implications for the new BASEL regulations that desire a high emphasis on market discipline, as well as the potential new regulations and laws that are expected in the aftermath of the current crisis.

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